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Publication date:
1996

Citation for published version (APA):
Berg, M., De Waegenaere, A. M. B., & Wielhouwer, J. L. (1996). *Optimal Tax Reduction by Depreciation: A Stochastic Model*. (CentER Discussion Paper; Vol. 1996-102). Accounting.

Optimal Tax Reduction by Depreciation: A stochastic model

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October 30, 1996

Abstract: This paper focuses on the choice of a depreciation method, when trying to minimize the expected value of the present value of future tax payments. In a quite general model that allows for stochastic future cash-flows and a tax structure with tax brackets, we determine the optimal choice between the straight line depreciation method and a specific accelerated depreciation method. We show how the distributions of the cash-flows, the discount rate, and the tax structure can influence the optimal decision. These results are illustrated by numerical examples. Contrarily to what is often assumed, the fact that future money is discounted does not necessarily imply that an accelerated depreciation method is preferable to a straight line method.

Keywords: minimizing expected tax payments, Straight line Depreciation Method (SDM), Accelerated Depreciation Method (ADM), uncertain cash-flows.

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1 Introduction

One of the accounting problems a firm faces is the decision on the depreciation method that will be used to represent the decrease in value over the years of a capital asset. Several methods of depreciation have been developed in the past[1], and the Financial Accounting Standards Board (FASB) gives some standards on which depreciation method a firm should use. In many cases however these standards leave some freedom for the firm to decide. Consequently, depreciation techniques are often used as a tool to gain financial benefits, rather than merely a tool to represent the decrease in value of the capital asset. Brief and Anton (1987) for example, argue that depreciation induces growth, i.e. there is a *multiplier effect.* More precisely, when an amount equal to depreciation is reinvested, a firm's productive capacity tends to increase, and when book depreciation is more accelerated than economic depreciation, reinvestment of depreciation increases a firm's financial capital.

The choice of a depreciation method clearly also has it's implications on the level of tax payments on future income, since the amounts depreciated influence the profit reported by the firm and consequently the total tax to be paid. Remer and Song (1993), Remer and Nieto (1993), and Jorgenson (1996) present empirical studies on the relation between taxes to be paid and depreciation method chosen. Davidson and Drake (1961, 1964), Roemmich et al. (1978), and Wakeman (1980) study the problem of minimizing the present value of tax payments by choosing the right depreciation method. In these papers however, the effect of future cash-flows on the optimal choice is ignored.

In this paper, we take stochastic future cash-flows into account, and compare the expected value of the present value of future tax payments for two depreciation methods: an accelerated depreciation method (ADM), and the straight line depreciation method (SDM). The ADM divides the depreciation charge into preset unequal parts that decrease from period to period, whereas SDM divides it into equal parts. It is customary to expect that due to the discounting effect, maximal earlier depreciation is advisable, so that ADM is preferable to SDM. While a lower discount rate will indeed always work in favor of ADM, it is demonstrated in Berg and Moore (1989) that relatively small cash-flows in the initial period can shift the preference to SDM. This, indeed, can only be true when there is separate taxing in each period, while if losses can be freely carried over from one period to another

[1] See for example Baxter (1971) for an overview on different depreciation methods.

then, as intuitively expected, ADM is unconditionally superior to SDM.

The aim of this paper is to carry further the issue of the choice between ADM and SDM by examining the effect on it of other factors of relevance, such as *the degree of uncertainty* involved in future cash-flows and the *level of progressiveness* in the tax structure. The focus here is on the stochastic cash-flows case and the setting is a general one: multi-period and multi-tax brackets with varying tax rates. We compare in these circumstances the present value of the tax paid over the periods when using ADM and SDM, and explore under what conditions one is preferable to the other. Since random variables are being compared an appropriate means of comparison is needed. In this paper we opt for the expected value comparison criterion, so that the depreciation method with the lower expected present value of tax payments is chosen. It is noteworthy that with this criterion possible dependencies among the cash-flow random variables in different periods are immaterial as far as the comparison here is concerned.

A summary of the main results of the paper is as follows:

- Increased uncertainty in the initial periods, which characterizes new product's riskiness, can work in favor of SDM. For example, in a situation where ADM is optimal, an increase in the cash-flow variance or a higher likelihood of low cash-flows in the first periods can shift the decision from ADM to SDM, even when the expected values of the cash-flows remain unchanged. Thus, taking into account the degree of uncertainty makes SDM the preferred method, which implies that by not considering the degree of uncertainty one can be led to the inferior ADM. In particular, if stochastic cash-flows are approximated by their deterministic expected values, as is often done, the conclusion can be opposite to what the more accurate stochastic analysis yields.

- Increased progressiveness of the tax structure in terms of tax brackets and/or corresponding tax rates, works in favor of SDM. Thus, in particular, even if ADM is optimal when only a fixed tax rate exists, a superimposed progressive tax structure can turn things around and make SDM optimal.

- Conveniently, given the cash-flow distributions and the tax structure, the quantitative choice rule between ADM and SDM is either trivial, or of the control-limit type. In the latter case, there exists a critical value $\tilde{\alpha}$ such that if the discount factor is below (resp. above) $\tilde{\alpha}$, then ADM (resp. SDM) is preferable. The critical value depends on the cash-flow distributions and on the tax structure parameters (i.e. brackets and corresponding rates). Note that this control-limit rule includes as a special case situations with "complete dominance of ADM over SDM", i.e. situations in which ADM is better than SDM for all[2] $\alpha < 1$ and at least as good as SDM for $\alpha = 1$, since this corresponds to a critical value $\tilde{\alpha} = 1$.

The paper is organized as follows. In section 2, we present the model and give a method to determine the optimal depreciation method for any given value of the model parameters. We further show that the choice rule is of the control-limit type, as explained above. In section 3, we study the effect of the variance of the distributions of the cash-flows (i.e. their riskiness) on the optimal decision. Section 4 studies the effect of the tax structure on the optimal decision. Section 5 deals with a tax system where carrying over of losses from one tax period to another is allowed. We show that in this case *ADM is universally better than SDM*. Section 6 concludes.

2 The model and the basic choice rule

Over a planning horizon of N periods, the decision maker has to determine the optimal depreciation method, where the alternatives are either the straight line depreciation method (SDM), or an accelerated depreciation method (ADM). The total depreciation charge over the N periods is denoted D. Consequently, under SDM, the depreciation charge in period i, $i = 1, ..., N$, equals $d = D/N$. Under ADM, the depreciation charge in period i is a given d_i, with $d_i \geq d_{i+1}$ for all $i = 1, 2, \ldots, N-1$, with at least one strict inequality, and $\sum_{i=1}^{N} d_i = D$.

At the end of each period, taxes have to be paid on the *reported income*, if positive. The reported income in period i equals the cash-flow C_i of period i minus the amount depreciated in period i. So, in period i taxes have to be paid on[3] $(C_i - d_i)^+$ in case of ADM, and on $(C_i - d)^+$ in case of SDM.

Many tax structures are progressive, i.e. up to a certain level K, one pays a tax rate T and above that level a higher tax rate is charged over the reported income above K. We consider m tax brackets. The levels above

[2]Discount rates $\alpha > 1$ are not considered here, since they correspond to negative interest rates. All the results in this paper however can easily be extended to the case where discount rates $\alpha > 1$ are taken into account.

[3]For a random variable X, one has $X^+ := \begin{cases} X & \text{if } X \geq 0 \\ 0 & \text{if } X \leq 0. \end{cases}$

which an extra charge is accounted for are denoted $K_1 < K_2 < ... < K_m$. Let $T_1 \leq T_2 \leq \ldots \leq T_{m-1}$ denote the tax rates that are charged over all reported income in the intervals $[K_j, K_{j+1}), j = 1, 2, \ldots, m-1$ respectively. Let $T_m \geq T_{m-1}$ denote the tax rate that is charged over all reported income above K_m. No taxes are charged on the reported income below K_1, so if $K_1 > 0$, then there is a tax-free bracket $[0, K_1)$. In the sequel, a tax structure $(m, (T_1, T_2, \ldots, T_m), (K_1, K_2, \ldots, K_m))$ will be abbreviated (m, T, K). Notice that a tax structure with a fixed tax rate T_1 over all reported income is a special case of the above by taking $(m, T, K) = (1, T_1, 0)$. Finally, the discount rate used to determine the present value of future money is denoted α, and for mathematical convenience, we define $T_0 := 0$. Then, given the above notations, the present value A of the total tax to be paid for the accelerated depreciation method is given by:

$$\begin{aligned} A &= \sum_{j=1}^{m-1} T_j \sum_{i=1}^{N} \alpha^{i-1}((C_i - d_i - K_j)^+ - (C_i - d_i - K_{j+1})^+) \\ &\qquad + T_m \sum_{i=1}^{N} \alpha^{i-1}(C_i - d_i - K_m)^+ \\ &= \sum_{j=1}^{m}(T_j - T_{j-1}) \sum_{i=1}^{N} \alpha^{i-1}(C_i - d_i - K_j)^+. \end{aligned} \tag{1}$$

Equivalently, the present value S of the total tax to be paid for the straight line depreciation method is given by:

$$S = \sum_{j=1}^{m}(T_j - T_{j-1}) \sum_{i=1}^{N} \alpha^{i-1}(C_i - d - K_j)^+. \tag{2}$$

The firm wants to choose the depreciation method that "minimizes" the present value of the total tax payments over the N periods. Since future cash-flows are usually not known with certainty, the $C_i, i = 1, 2, \ldots, N$ are random variables[4]. So, to determine which depreciation method should be used, one has to compare two random variables, A and S. As explained in the introduction, we opt for a risk-neutral approach, and consequently compare the random variables by means of their expected value. Let us denote by $F_i(.)$ the distribution function of C_i. Since, for any random variable X with distribution function $F(.)$, and any constant c, one has

$$E[(X - c)^+] = \int_c^\infty (1 - F(u))du,$$

[4] Notice that the deterministic case is a special case of this model, because cash-flows that are known with certainty correspond to degenerate distribution functions.

taking the expected value of A and S yields:

$$E[A] = \sum_{j=1}^{m}(T_j - T_{j-1})\sum_{i=1}^{N}\alpha^{i-1}\int_{d_i+K_j}^{\infty}(1-F_i(u))du, \qquad (3)$$

$$E[S] = \sum_{j=1}^{m}(T_j - T_{j-1})\sum_{i=1}^{N}\alpha^{i-1}\int_{d+K_j}^{\infty}(1-F_i(u))du. \qquad (4)$$

Now, a risk-neutral decision maker will prefer ADM to SDM (resp. SDM to ADM) if $E[A] < E[S]$ (resp. $E[A] > E[S]$). He is indifferent between the two methods if $E[A] = E[S]$.

Throughout this section, we will assume that the number of periods N, the total amount to depreciate D, the depreciation charges $d_i, i = 1, 2, \ldots, N$ for ADM, as well as the distribution functions for the cash-flows $F_i(.)$, $i = 1, 2, \ldots, N$, and the tax structure (m, T, K) are given. Consequently, $E[A]$ and $E[S]$ are functions of the discount rate α, and we denote by $E[A|\alpha]$ and $E[S|\alpha]$, the respective expected values given a discount rate α.

We now state the main result of this section, namely that the optimal choice between ADM and SDM is of the control-limit type, i.e. there exists a critical value of the discount rate α below which ADM is preferable and above which, SDM is preferable.

Proposition 2.1: *Either $E[A|\alpha] = E[S|\alpha] = 0$ for all α, or there exists an $\tilde{\alpha} \le 1$ such that ADM is preferable to SDM for all $\alpha \in [0, \tilde{\alpha})$, and SDM is preferable to ADM for all $\alpha \in (\tilde{\alpha}, 1]$.*

Proof: For notational convenience, we define the following function:

$$g(\alpha) := \sum_{j=1}^{m}(T_j - T_{j-1})\sum_{i=1}^{N}\alpha^{i-1}\int_{d_i+K_j}^{d+K_j}(1-F_i(u))du. \qquad (5)$$

Then, (3) and (4) imply that for any given discount rate α, one has $E[A|\alpha] - E[S|\alpha] = g(\alpha)$. Consequently, we need to show that either $E[A|\alpha] = E[S|\alpha] = 0$ for all α, or there exists a value $\tilde{\alpha} \le 1$ such that $g(\alpha) < 0$ for all $\alpha \in [0, \tilde{\alpha})$ and $g(\alpha) > 0$ for all $\alpha \in (\tilde{\alpha}, 1]$.

It is clear that ADM implies that $d_1 > d$, and we thus have that $g(0) \le 0$. In Appendix A, we show that either $g(\alpha) = 0$ for all α, or there is at most one $\alpha \ge 0$ satisfying $g(\alpha) = 0$. Then clearly, since $g(.)$ is continuous, there are three possibilities:

- $g(\alpha) = 0$ for all α. This implies that for all $i = 1, 2, \ldots, N$, one has:

$$\sum_{j=1}^{m}(T_j - T_{j-1}) \int_{d+K_j}^{d_i+K_j} (1 - F_i(u))du = 0$$

  Since $T_j - T_{j-1} \geq 0$, and $1 - F_i(.) \geq 0$, this implies that $P(C_i \leq \min(d_i + K_j, d + K_j)) = 1$, for all periods $i = 1, 2, \ldots, N$, and all brackets $j = 1, 2, \ldots, m$. Consequently in this case, one has $E[A|\alpha] = E[S|\alpha] = 0$ for all α;

- The equation $g(\alpha) = 0$ has no solutions in $[0, 1]$. Then, since $g(0) \leq 0$, it follows that $g(\alpha) < 0$ for all $\alpha \in [0, 1]$, so one can take $\tilde{\alpha} = 1$.

- The equation $g(\alpha) = 0$ has a unique solution $\tilde{\alpha}$ in $[0, 1]$. Then, since $g(0) \leq 0$, it follows that $g(\alpha) < 0$ for all $\alpha \in [0, \tilde{\alpha})$, and $g(\alpha) > 0$ for all $\alpha \in (\tilde{\alpha}, 1]$, and $g(\tilde{\alpha}) = 0$.

This completes the proof. $\square$

The discount rate $\tilde{\alpha}$ appearing in proposition 2.1 will be called in the sequel the *critical value*. Notice that $E[A|\alpha] = E[S|\alpha] = 0$ for all α if and only if $P(C_i \leq \min(d_i + K_j, d + K_j)) = 1$, for all periods $i = 1, 2, \ldots, N$, and all brackets $j = 1, 2, \ldots, m$. In this case, $(C_i - d)^+$ and $(C_i - d)^+$ are, with probability one, equal to zero if $K_1 = 0$, or in the tax-free bracket if $K_1 > 0$. Then clearly the problem becomes a trivial one, since no taxes will have to be paid, with probability one, for both methods. So the above proposition says that in all cases for which the problem is not the trivial one described above, a critical value exists.

Remarks:

- In order to know for a given value of α which method is optimal, one simply has to verify the sign of $g(\alpha)$.

- When a critical value $\tilde{\alpha} < 1$ exists, it follows immediately from the definition of the critical value that $g(\tilde{\alpha}) = 0$, i.e. ADM and SDM are equally good when α equals the critical value. When $\tilde{\alpha} = 1$, ADM is preferable to SDM for all $\alpha < 1$, and ADM is at least as good as SDM for $\alpha = 1$ $(g(1) \leq 0)$.

We conclude this section with an example. It illustrates how the choice between ADM and SDM changes with the value of α. The intuition that more discounting (a lower α) works in favor of ADM, is confirmed.

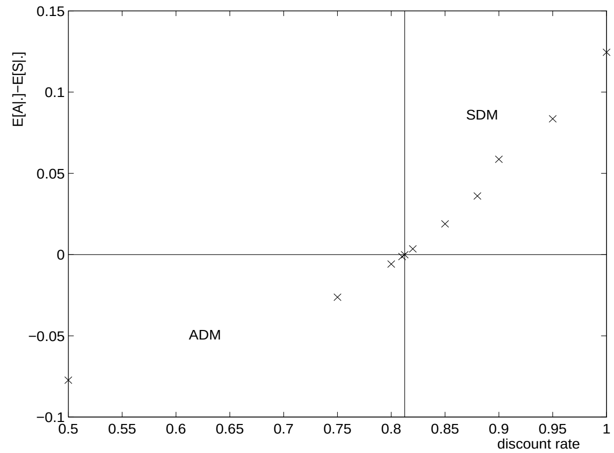

Figure 1: The effect of α on the optimal choice between ADM and SDM.

Example 2.1: Consider the case where $D = 5$, and $N = 5$. Consequently, for SDM, one has $d = 1$. The depreciation charges for ADM are: $d_1 = 1.8, d_2 = 1.4, d_3 = 1, d_4 = 0.6$, and $d_5 = 0.2$. The cash-flows are normally distributed with for the five subsequent periods : $C_1 \sim N(1,1)$, $C_2 \sim N(3,2)$, $C_3 \sim N(5,3)$, $C_4 \sim N(4,3)$, and $C_5 \sim N(2,2)$. The tax structure is given by $(m,T,K) = (3,(0.2,0.3,0.4),(0,2,4))$, i.e. there are three tax brackets $[0,2),[2,4)$ and $[4,+\infty)$ with tax rates $0.2, 0.3$ and 0.4 respectively. In figure 1, $g(\alpha) = E[A|\alpha] - E[S|\alpha]$ is plotted for different values of the discount rate α. It is easily verified that the critical value $\tilde{\alpha}$ equals 0.8127, i.e. for all $\alpha > 0.8127$, SDM is optimal, for all $\alpha < 0.8127$, ADM is optimal, and for $\alpha = 0.8127$, both methods are equally good. $\square$

3 Effect of the cash-flow distributions on the choice of the depreciation method

The critical value $\tilde{\alpha}$ that determines the optimal choice between ADM and SDM depends on the cash-flow distributions and in this section we want to gain more insight into the nature of this dependence. We assume that the tax structure (m,T,K) is given.

It is important to notice that the choice between ADM and SDM is nontrivial because of the uncertainty of future cash-flows. In contrast, if future cash-flows were known with certainty, then the optimal choice follows immediately by straightforward calculations. So, a decision maker might approximate the stochastic future cash-flows by their expected values, and then perform the (trivial) calculations for the deterministic case. In this case, the variance (i.e. the riskiness) of the realizations of future cash-flows is ignored. Intuitively, one might expect however that when earlier cash-flows become more uncertain (i.e. higher variance), then, even when expected cash-flows did not change, the optimal depreciation method can shift from ADM to SDM because the probability of having a negative reported income in the early periods becomes higher. Thus, ignoring the variance of the cash-flows, can lead to non-optimal decisions. This is illustrated in the following example where it is shown that a change in the variances of the cash-flow distributions can change the choice of the optimal depreciation method, even if the expected values of the future cash-flows remain unchanged.

Example 3.1: Consider the case where $D = 8$, and $N = 5$. Consequently, for SDM one has $d = 1.6$. The depreciation charges for ADM are: $d_1 = 3, d_2 = 2.3, d_3 = 1.6, d_4 = 0.9$ and $d_5 = 0.2$. The cash-flows are normally distributed with, for the five subsequent periods, $C_i \sim N(4, 1)$. The discount rate equals 0.95. There is a fixed tax rate $T_1 = 0.2$ over all reported income, i.e. $(m, T, K) = (1, 0.2, 0)$. We find that $E[A|\alpha = 0.95] - E[S|\alpha = 0.95] = g(0.95) = -0.047 < 0$. Consequently, ADM is optimal. Notice also that when cash-flows are deterministic, i.e. $C_i = 4$, ADM is clearly the optimal decision.
Let us now suppose that the decision maker acknowledges that although the *expected* cash-flow still equals 4 in each of the periods, the risk of having a lower cash-flow in the first two periods is higher than in the above situation. More precisely, the standard deviation in the first two periods is estimated to be equal to 2 instead of 1. One then finds that $g(0.95) = 0.011 > 0$, and consequently, SDM is optimal. □

The above example illustrates that, in a situation where ADM is optimal, an increase in the variance in the first periods can shift the choice to SDM. The underlying reason is that the increase in the variance of the distribution of the cash-flow implies that the probability of having a cash-flow in period one (resp. period two) which is lower than d_1 (resp. d_2), becomes higher. In that case it is better not to depreciate too much in the first periods. This is confirmed in the following proposition. It essentially states that when there is a higher probability of realizing "low" cash-flows in the early periods, the optimal decision may shift from ADM to SDM.

Let us denote by $k \in \{1, 2, \ldots, N\}$ the last period in which the depreciation charge for ADM exceeds the charge for SDM, i.e, $d_i > d$ for $i = 1, 2, \ldots, k$, and $d_i \leq d$ for $i = k + 1, \ldots, N$. We now introduce the following definition.

Definition 3.1: *Distribution functions $G_1(.), G_2(.), \ldots, G_k(.)$ are "cash-flow riskier" than distribution functions $F_1(.), F_2(.), \ldots, F_k(.)$ if for each $i \in \{1, 2, \ldots, k\}$ one has $G_i(x) \geq F_i(x)$ for all $x \leq d_i + K_m$, and for at least one $i \leq k$ and $j \leq m$, there exist a and b such that $a < b \in [d + K_j, d_i + K_j]$ and $G_i(x) > F_i(x)$ for all $x \in (a, b)$.*

We then have the following result:

Proposition 3.1: *Suppose that $F_i(.), i = 1, 2, \ldots, N$ are cash-flow distributions for which a critical value $\tilde{\alpha} \in (0, 1)$ exists. Suppose furthermore that $G_i(.), i = 1, 2, \ldots, k$ are cash-flow riskier distributions than $F_i(.), i = 1, 2, \ldots, N$. Then there exists a discount rate $\tilde{\alpha}_{cfr} < \tilde{\alpha}$ such that for all discount rates $\alpha \in (\tilde{\alpha}_{cfr}, \tilde{\alpha})$, SDM is optimal for cash-flow distribution functions $G_1(.), \ldots, G_k(.), F_{k+1}(.), \ldots, F_N(.)$, whereas ADM is optimal for cash-flow distribution functions $F_1(.), F_2(.), \ldots, F_N(.)$.*

Proof: For simplicity of notation, we consider the case in which there is a fixed tax rate $T_1 > 0$, i.e. $(m, T, K) = (1, T_1, 0)$, and $G_i(.) = F_i(.)$ for $i = 2, \ldots, k$, i.e. only the distribution of C_1 is considered to be cash-flow riskier than $F_1(.)$. In this case, the function $g(.)$ defined in (5) simplifies to $g(\alpha) := T_1 \sum_{i=1}^{N} \alpha^{i-1} \int_{d_i}^{d} (1 - F_i(u))du$. Since $g(.)$ is continuous, and $\tilde{\alpha} < 1$, it follows that $g(\tilde{\alpha}) = 0$. Let us now define the function:

$$h(\alpha) := T_1 \int_{d_1}^{d} (1 - G_1(u))du + T_1 \sum_{i=2}^{N} \alpha^{i-1} \int_{d_i}^{d} (1 - F_i(u))du.$$

Then ADM (resp. SDM) is optimal for the new distributions $G_1(.), F_2(.), \ldots, F_N(.)$ if $h(\alpha) < 0$ (resp. > 0). We define

$$\tilde{\alpha}_{cfr} := \inf \{\alpha \geq 0 \mid h(\alpha) > 0\}.$$

Now, since there exist a and b such that $d \leq a < b \leq d_1$ and $G_1(x) > F_1(x)$ for all $x \in (a, b)$, it follows that $\int_{d}^{d_1} (1 - G_1(u))du < \int_{d}^{d_1} (1 - F_1(u))du$. Therefore, one has $h(\tilde{\alpha}) > g(\tilde{\alpha}) = 0$. It then follows from the definition of $\tilde{\alpha}_{cfr}$ that $0 \leq \tilde{\alpha}_{cfr} < \tilde{\alpha} < 1$. Then, it follows from proposition 2.1 that $\tilde{\alpha}_{cfr}$ is the critical value for the cash-flow riskier distributions. Consequently, for these distributions, ADM is preferable to SDM for all $\alpha \in (0, \tilde{\alpha}_{cfr})$, and SDM is preferable to ADM for all $\alpha \in (\tilde{\alpha}_{cfr}, 1)$. This yields the desired result. □

Let us now return to example 3.1.

Example 3.1 (continued): Denote $F(.)$ for the distribution function of an $N(4, 1)$ random variable, and $G(.)$ for the distribution function of an

$N(4,2)$ random variable. Then in the first situation in example 3.1, one has $F_i(.) = F(.)$ for $i = 1, 2, \ldots, 5$, and in the second situation one has $F_1(.) = F_2(.) = G(.)$ and $F_3(.) = F_4(.) = F_5(.) = F(.)$. Now it is easy to show that $G(x) > F(x)$ for all $x \leq d_1$. Consequently, since $k = 2$ and $d_2 \leq d_1$, the distribution functions in the second situation are cash-flow riskier than the distribution functions in the first situation. Furthermore, it is easy to show that, with the notation introduced in proposition 3.1, for $F_i(.) = F(.), i = 1, 2, \ldots, 5$, one has $\tilde{\alpha} \approx 0.987$, and for $F_1(.) = F_2(.) = G(.)$ and $F_3(.) = F_4(.) = F_5(.) = F(.)$, one has $\tilde{\alpha}_{cfr} \approx 0.942$. Therefore, for all $\alpha \in (0.942, 0.987)$, ADM is optimal in the first situation, but SDM is optimal in the second (cash-flow riskier) situation.

4 Effect of the tax structure on the choice of the depreciation method

The critical value $\tilde{\alpha}$ that determines the optimal choice between ADM and SDM depends also on the tax structure. We now want to gain more insight into the nature of this dependence. For clarity of exposition we therefore denote here the critical value corresponding to a tax structure (m, T, K) by $\tilde{\alpha}(m, T, K)$. Consider now two tax structures: (m, T, K) and $(\bar{m}, \bar{K}, \bar{T})$. Since in general $\tilde{\alpha}(m, T, K) \neq \tilde{\alpha}(\bar{m}, \bar{T}, \bar{K})$, and because of the control-limit nature of the choice between ADM and SDM, the optimal choice is different for any discount rate α between $\tilde{\alpha}(m, T, K)$ and $\tilde{\alpha}(\bar{m}, \bar{T}, \bar{K})$. This is illustrated in the following example.

Example 4.1: Consider the case where $D = 7$, and $N = 5$. Consequently, for SDM, one has $d = 1.4$. The depreciation charges for ADM are: $d_1 = 2, d_2 = 1.75, d_3 = 1.5, d_4 = 1$, and $d_5 = 0.75$. The cash-flows are normally distributed with distributions $C_1 \sim N(2.5, 1), C_2 \sim N(3, 1.5), C_3 \sim N(5, 2), C_4 \sim N(5, 2)$, and $C_5 \sim N(5, 2)$. First we consider the case with a fixed tax rate $T_1 = 0.2$ over all reported income, i.e. $(m, T, K) = (1, 0.2, 0)$. On the other hand, we consider a tax structure with the same parameters as before but with an extra tax rate of 0.1 for all reported income above 3, i.e. $(\bar{m}, \bar{T}, \bar{K}) = (2, (0.2, 0.3), (0, 3))$. We find that $\tilde{\alpha}(m, T, K) \approx 0.945$ and $\tilde{\alpha}(\bar{m}, \bar{T}, \bar{K}) \approx 0.875$. This implies that for all discount rates $\alpha \in (0.875, 0.945)$, ADM is optimal for the first tax structure, but the introduction of an extra tax bracket makes SDM optimal.
Let us look at the above situation again, but now suppose that $\alpha = 0.9$ is given. Then the following question arises: Does a higher level of the tax rate T_2 for the new bracket $[3, +\infty)$ favor SDM, and if so, what level of T_2 makes

the decision switch from ADM to SDM? For $\alpha = 0.9$, $m = 2, T_1 = 0.2$ and $K = (0, 3)$, we see that for any value of T_2, one has $g(0.9) = -0.096 + 0.39T_2$, i.e. the coefficient in $g(0.9)$ of T_2 is positive. Consequently, a higher level of T_2 favors SDM. Furthermore, we see that SDM becomes optimal when $g(0.9) > 0$, i.e. when $T_2 > \bar{T}_2 = 0.246$. □

The intuition behind this example is the fact that since the expected cash-flows are increasing, the probability of getting into the new tax bracket $[3, +\infty)$ in later periods is higher than in early periods. Therefore, in the presence of this higher tax bracket, it is better not to depreciate too much in the first periods. This effect of course gets stronger as the tax rate in that new bracket gets higher. This result is formalized in the following proposition. It gives a condition under which the probabilities of getting into a new tax bracket $[K_{m+1}, +\infty)$ in later periods ($i \geq k+1$) are high enough (relative to the probabilities of getting into that tax bracket in early periods ($i \leq k$)) in order to compensate the discounting effect. Under this condition, the introduction of this extra bracket favors SDM.

Proposition 4.1: *Let the discount rate $\alpha \in [0, 1]$, cash-flow distributions $F_1(.), \ldots, F_N(.)$, and tax structure $(m, (T_1, \ldots, T_m), (K_1, \ldots, K_m))$ be given. If $K_{m+1} > K_m$ is such that:*

$$\sum_{i=1}^{k} \alpha^{i-1}(d_i - d)P(C_i - d > K_{m+1}) < \sum_{i=k+1}^{N} \alpha^{i-1}(d - d_i)P(C_i - d > K_{m+1}), \tag{6}$$

then there exists a $\bar{T}_{m+1}$ such that SDM is optimal for all tax structures $(m+1, (T_1, \ldots, T_m, T_{m+1}), (K_1, \ldots, K_m, K_{m+1}))$ with $T_{m+1} > \bar{T}_{m+1}$.

Proof: Consider, for given $T_{m+1} > T_m$ and $K_{m+1} > K_m$, the tax structure $(m+1, (T_1, \ldots, T_m, T_{m+1}), (K_1, \ldots, K_m, K_{m+1}))$, and let $g(.)$ denote the corresponding polynomial, as defined in (5). It is seen easily that for this tax structure, the difference in reported income above K_{m+1} between ADM and SDM, i.e. the coefficient of T_{m+1} in $g(.)$, is given by:

$$\sum_{i=1}^{N} \alpha^{i-1} \int_{d_i + K_{m+1}}^{d + K_{m+1}} (1 - F_i(u))du. \tag{7}$$

Therefore, when (7) is positive, an increase in T_{m+1} works in favor of SDM. Since (6) clearly implies that (7) is positive, the proof is concluded. □

It is noteworthy that for all $\alpha \in [0, 1]$, one has (see lemma 5.1 in section 5 for a proof):

$$\sum_{i=1}^{k} \alpha^{i-1}(d_i - d) \geq \sum_{i=k+1}^{m} \alpha^{i-1}(d - d_i).$$

Therefore, (6) can only be true if the probability of getting a reported income $C_i - d$ above K_{m+1} for later periods ($i > k$) is higher than the probability of getting a reported income $C_i - d$ above K_{m+1} for early periods ($i \leq k$).

5 Carrying over of losses

In this section, we study the optimal choice between ADM and SDM over a number N of periods in a tax system where carrying over of losses is allowed. Within this tax system, we again denote a tax structure, i.e. levels, brackets and rates, by (m, T, K). Consequently, one has for the present value of total tax payments under ADM and SDM respectively:

$$\begin{aligned} A &= \sum_{j=1}^{m}(T_j - T_{j-1})[\sum_{i=1}^{N} \alpha^{i-1}(C_i - d_i) - K_j]^+, & (8) \\ S &= \sum_{j=1}^{m}(T_j - T_{j-1})[\sum_{i=1}^{N} \alpha^{i-1}(C_i - d) - K_j]^+. & (9) \end{aligned}$$

In the previous sections we saw that, for given depreciation charges, cash-flow distributions, and tax structure, the optimal choice between ADM and SDM essentially depends on the discount factor α. We also studied how the tax structure and the distributions of the cash-flows can influence the optimal decision. The following lemma shows that this is no longer the case in the tax system considered in this section, where ADM is *universally better* than SDM.

Lemma 5.1: *For all discount rates* $\alpha \in [0, 1]$, *tax structures* (m, T, K), *and distribution functions* $F_i(.), i = 1, 2, \ldots, N$, *and for every choice of* $d_i \geq d_{i+1}, i = 1, 2, \ldots, N-1$ *with at least one strict inequality and satisfying* $\sum_{i=1}^{N} d_i = D$, *ADM is at least as good as SDM.*

Proof: For notational convenience, we give the proof for the case where $m = 1$. First notice that in order to have $E[A] \leq E[S]$, it is sufficient that $A \leq S$, which is, for a given value of α, clearly satisfied when $\sum_{i=1}^{N} \alpha^{i-1} d_i \geq d \sum_{i=1}^{N} \alpha^{i-1}$. Therefore, it is sufficient to show that $\sum_{i=1}^{N} \alpha^{i-1}(d_i - d) \geq 0$

for all $\alpha \in [0,1]$. Now $\sum_{i=1}^{N} d_i = Nd$ implies that for all $j = 1, 2, \ldots, N$:

$$\sum_{i=1}^{j}(d_i - d) = \sum_{i=j+1}^{N}(d - d_i) = -\sum_{i=j+1}^{N}(d_i - d).$$

Let us denote $k := \max\{i \in \{1, 2, \ldots, N\} \mid d_i > d\}$. It then follows that:

$$\begin{aligned}\sum_{i=1}^{k} \alpha^{i-1}(d_i - d) &\geq \alpha^k \sum_{i=1}^{k}(d_i - d) \\ &= -\alpha^k \sum_{i=k+1}^{N}(d_i - d) \\ &\geq -\sum_{i=k+1}^{N} \alpha^{i-1}(d_i - d).\end{aligned}$$

Therefore $\sum_{i=1}^{N} \alpha^{i-1}(d_i - d) \geq 0$. Furthermore, this inequality is strict for all $\alpha \in [0,1)$. This concludes the proof. □

6 Conclusions

In making the decision on the choice of a depreciation method, many factors are important. This paper focused on the issue of minimizing the present value of total tax payments over a number of periods, when a choice has to be made between the straight line depreciation method and an accelerated depreciation method. It is shown how the discount rate, the tax structure, and the distributions of the future cash-flows affect the optimal decision. Sensitivity analysis with respect to these parameters is performed and illustrated in numerical examples. For future research, one could study an extension to a situation where different investment possibilities at the different periods can influence the optimal decision. Furthermore, one could incorporate the problem into a strategic framework. We showed how the tax structure influences the optimal decision. Therefore, it might be interesting to consider a "game", where the players are the tax authority on one hand, and the firms on the other hand. Anticipating optimal behavior by the firms the tax authority might want to have a tax structure that, within certain limits, maximizes its income.

Appendix A

In this appendix, we show that, for the function $g(.)$ defined in (5) either $g(\alpha) = 0$ for all α, or there is at most one $\alpha \geq 0$ satisfying $g(\alpha) = 0$.

For simplicity of notation, we give the proof for the case where there is a fixed tax rate equal to 1, i.e. $(m, T, K) = (1, 1, 0)$. Let us again use the following notation:

$$k := \max\{i \in \{1, \ldots, N\} \mid d_i > d\}.$$

Taking the derivative of $g(.)$ with respect to α, and keeping in mind the definition of k, yields for $\alpha > 0$:

$$\begin{aligned}
g'(\alpha) &= \frac{\partial}{\partial \alpha}\left(-\sum_{i=1}^{k} \alpha^{i-1} \int_{d}^{d_i} (1 - F_i(u))du + \sum_{i=k+1}^{N} \alpha^{i-1} \int_{d_i}^{d} (1 - F_i(u))du\right) \\
&= -\sum_{i=2}^{k} (i-1)\alpha^{i-2} \int_{d}^{d_i} (1 - F_i(u))du + \sum_{i=k+1}^{N} (i-1)\alpha^{i-2} \int_{d_i}^{d} (1 - F_i(u))du \\
&\geq (k-1)\left[-\sum_{i=2}^{k} \alpha^{i-2} \int_{d}^{d_i} (1 - F_i(u))du + \sum_{i=k+1}^{N} \alpha^{i-2} \int_{d_i}^{d} (1 - F_i(u))du\right] \\
&= (k-1)\left[\sum_{i=2}^{N} \alpha^{i-2} \int_{d_i}^{d} (1 - F_i(u))du\right] \\
&= (k-1)\left[g(\alpha)/\alpha + \int_{d}^{d_1} (1 - F_1(u))du\right].
\end{aligned}$$

Since $d_1 > d$, it follows that $g'(\alpha)$ is non-negative whenever $g(\alpha)$ is non-negative.

We now consider two cases:

- $d_2 \geq d$: Then it can be shown as above that $g''(\alpha)$ is non-negative whenever $g'(\alpha) = 0$. Consequently, $g(.)$ does not have local maxima.
- $d_2 < d$: Then it follows that $k = 1$. In this case, the above implies that $g'(\alpha) \geq 0$ for all $\alpha \geq 0$.

In both cases, since $g(.) = 0$ is a polynomial equation, this implies that either $g(\alpha) = 0$ for all α, or there is at most one $\alpha \geq 0$ satisfying $g(\alpha) = 0$. This concludes the proof. $\square$

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